# SOLUTION OF THE PROBLEM OF THE GENERAL CASE OF TWO-DIMENSIONAL GRANULE (BODY*) MOTION IN A GRAVITY FIELD

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Analytical dependences governing the motion parameters are obtained for the two-dimensional problem of the motion of granules ejected at an arbitrary angle to the horizon in an obliquely directed stream with a plane velocity profile.

The need to determine the parameters of two-dimensional motion of bodies ejected at an angle to the horizon originates in a number of practical cases. The motion of the ambient medium hence often turns out to be obliquely directed.

Such a situation can hold, for instance, in the separate stages of the tower granulation process.

As is easy to determine, the soaring velocities of granules of 1-3 mm size are on the order of 6-12 m/sec, and the Reynolds numbers are hence on the order of 500-2300. It is known that the magnitude of the frontal drag coefficient in the flow around a sphere in this Re number range can be considered constant to sufficient accuracy. Air supplied from below into the operating towers has a 0.3-0.6 m/sec velocity (these magnitudes reach 1.5-1.7 m/sec only in individual structures). There hence follows that the kinematics of air flow around granules should be determined by the air-motion velocity, since the fluctuating air-velocity components (the averaged values) have values, in practice, which are substantially less than the mean discharge velocity which is approximately an order of magnitude lower than the velocity of granule motion.

In an examination of bodies falling in a gravity field, the papers [1-3] do not yield final solutions to determine the fundamental body-motion parameters in an obliquely directed flow in a general formulation of the problem.

Let us examine the general case of a body with a constant drag coefficient dropping in a gravity field (turbulent flow with a laminar boundary layer). We take as the initial system of two-dimensional motion equations

$$\left.\begin{aligned} m\frac{d^2x}{d\tau^2} &= \left(-\frac{1}{2}\rho\zeta F w^2\right)_x, \\ m\frac{d^2y}{d\tau^2} &= \left(-\frac{1}{2}\rho\zeta F w^2\right)_y + mg \end{aligned}\right\}. \tag{1}$$

Let us represent the system (1) in dimensionless form. To do this we introduce the following characteristic variables by using the results from [3]:

$$V=\frac{v}{w_\infty};\quad U=\frac{u}{w_\infty};\quad W=\frac{w}{w_\infty};\quad \theta=\frac{\tau}{\tau_\infty};\quad X=\frac{x}{g\tau_\infty^2};\quad Y=\frac{y}{g\tau_\infty^2},$$

*Bodies with a size on the order of 1 mm and greater whose soaring velocities correspond to a turbulent flow mode are kept in mind.

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where $w_\infty = \sqrt{g/a}$ is the soaring velocity, $a = (1/2m)\cdot\rho\xi F$;

$$\tau_\infty = \frac{w_\infty}{g}; \quad \bar{w} = \bar{v} \pm \bar{u} \quad \text{or} \quad \bar{W} = \bar{V} \pm \bar{U}.$$

Consequently, we obtain

$$\left.\begin{aligned} &\frac{d^2X}{d\theta^2} + WW_x = 0, \\ &\frac{d^2Y}{d\theta^2} + WW_y = 1 \end{aligned}\right\} \tag{1a}$$

or since

$$\frac{dX}{d\theta} = W_x \text{ and } \frac{dY}{d\theta} = W_y,$$

$$\left.\begin{aligned} &\frac{dW_x}{d\theta} + W_x^2 \sqrt{\left(\frac{W_y}{W_x}\right)^2 + 1} = 0, \\ &\frac{dW_y}{d\theta} \pm W_y^2 \sqrt{\left(\frac{W_x}{W_y}\right)^2 + 1} = 1 \end{aligned}\right\}. \tag{1b}$$

Here and henceforth, the upper sign will correspond to ejection of the particles at an angle $\alpha_0 \leq \pi/2$, and the lower to $-\alpha_0 > \pi/2$, where $\alpha_0$ is an angle measured from a downwardly directed vertical.*

The relationship

$$d \ln \frac{W_y}{W_x} = \frac{d\theta}{W_y} \tag{2}$$

can be obtained from the system (1a).

An exact analytical solution of the system (1b) cannot be performed successfully. However, in contrast to ballistics problems which require high accuracy of the solution, approximate solutions are completely acceptable in the examination of engineering questions. Such an approximate solution has been performed in [3], for example, when the ratio $W_x/W_y$ in (1b) was considered negligible in comparison to one. However, it turned out to be possible to carry out a broader, practically complete solution of the question. To do this, we assume that the ratio $W_y/W_x$ in the first equation in the system (1b) is determined by means of the second equation in which the right side is taken for the case $W_x/W_y \ll 1$. This latter assumption will reflect reality more exactly, the smaller the ratio $W_x/W_y$, and, conversely, the deviation will grow as the ratio $W_x/W_y$ increases.

On the other hand, the ratio $W_y/W_x$ which will be smaller, the greater the ratio $W_x/W_y$, is used in the first equation of the system (1b).

Therefore, although its definition by the above-mentioned approximate method will result, as $W_x/W_y$ increases, in an increase in the discrepancy between the $W_x/W_y$ values obtained and the real values, nevertheless, the role of the first member under the radical in the first equation of (1b) will hence drop rapidly. Conversely, when $W_y/W_x$ increases in the latter and the role of this member grows, then $W_x/W_y$ drops, and the accuracy of determining this ratio also grows.

Taking the above into account, it is quite expedient to solve the system (1b) on the basis of the assumption mentioned.

Let us express $d\theta$ in terms of $dW_y$ from the second equation of the system (1b) for $(W_x/W_y)^2 \ll 1$ and by substituting this expression in the right side of (2) we obtain

*For the case $\alpha_0 = \pi/2$ the spoiling of the assumption made about the constancy of the frontal drag coefficient is possible. However, it should be noted that under all technical conditions some value of the horizontal velocity component is conserved at the point where the vertical velocity component drops to zero. The total velocity can hence be reduced substantially and result in spoilage of the assumption about the constancy of the drag coefficient in individual cases. The considerations elucidated above refer, however, just to the domain directly adjacent to a point where the vertical component becomes zero and operate, correspondingly, on relatively small sections of the body-motion trajectory. In sum, the influence of this factor on the final results of the computations turns out to be insignificant.

$$d \ln \frac{W_y}{W_x} = \frac{dW_y}{W_y(1 \mp W_y^2)}. \tag{3}$$

Solving (3), we find

$$\left(\frac{W_y}{W_x}\right)^2 = \pm \frac{1}{W_x^2} \mp \frac{1 \mp W_{y0}^2}{W_{x0}^2}, \tag{4}$$

where the subscript "0" corresponds to initial conditions. Let us substitute (4) into the first equation of the system (1b),

$$\frac{dW_x}{d\theta} + W_x^2 \sqrt{\pm \frac{1}{W_x^2} \mp \frac{1 \mp W_{y0}^2 \mp W_{x0}^2}{W_{x0}^2}} = 0. \tag{5}$$

Taking $1/W_X$ as a new variable and solving (5), we obtain for $\alpha_0 \leq \pi/2$

$$W_x = \frac{W_{x0}}{\operatorname{ch}\theta + W_0 \operatorname{sh}\theta} \tag{6}$$

or, furthermore, introducing the notation

$$K = \sqrt{\frac{1 + W_0}{1 - W_0}},$$

$$W_x = \frac{2W_{x0}}{1 - W_0} \cdot \frac{\exp\theta}{K^2 \exp 2\theta + 1}, \tag{6a}$$

for $\alpha_0 > \pi/2$,

$$W_x = \frac{W_{x0}}{\sqrt{1 + W_0^2}} \cdot \frac{1}{\sin(\theta + \beta)}, \tag{6b}$$

where $\beta = \arcsin[1/\sqrt{1 + W_0^2}]$.

The computation using (6b) is carried out until the time when $W_y = 0$. When a further computation is needed, the point corresponding to $W_y = 0$ is taken as the origin, and a computation is performed from it by means of (6a) corresponding to $\alpha_0 \leq \pi/2$.

An analogous method of computations for the case $\alpha_0 > \pi/2$ is taken in seeking the other particle motion parameters by means of the relationships given below.

Substituting $dX/d\theta = V_X = W_X \mp U_X$ and integrating, we find ($\theta = 0$ for $X = 0$) for $\alpha_0 \leq \pi/2$

$$X = \frac{2W_{x0}}{\sqrt{1 - W_0^2}} \operatorname{arctg}\left(K \frac{\exp\theta - 1}{K^2 \exp\theta + 1}\right) \mp U_x\theta, \tag{7}$$

where the factors in front of the arctan and the ln in (7) turn out to be imaginary for $W_0 > 1$.

Then taking account of the known relationship between the arctan and the ln, we obtain in the complex domain

$$X = \frac{W_{x0}}{\sqrt{W_0^2 - 1}} \ln \frac{1 - i/K}{1 + i/K} \cdot \frac{\exp\theta + i/K}{\exp\theta - i/K} \mp U_x\theta, \tag{7a}$$

for $\alpha_0 > \pi/2$

$$X = \frac{W_{x0}}{\sqrt{1 + W_0^2}} \ln\left[\operatorname{tg}\frac{\theta + \beta}{2}\left(\sqrt{1 + W_0^2} + W_0\right)\right] \mp U_x\theta. \tag{7b}$$

To seek $W_y$ and Y, let us substitute the (6) and (6a) found into (2):

for $\alpha_0 \leq \pi/2$

$$\frac{dW_y}{d\theta} + \frac{K^2 \exp 2\theta - 1}{K^2 \exp 2\theta + 1} W_y = 1, \tag{8}$$

for $\alpha_0 > \pi/2$

$$\frac{dW_y}{d\theta} + W_y \operatorname{ctg}(\theta + \beta) = 1. \tag{8a}$$

The solution of these equations will be:

for $\alpha_0 \leq \pi/2$

$$W_y = \frac{\exp\theta}{K^2\exp 2\theta + 1}\,[W_{y0}(K^2+1) - (K^2-1) + K^2\exp\theta - \exp(-\theta)], \tag{9}$$

for $\alpha_0 > \pi/2$

$$W_y = -\operatorname{ctg}(\theta+\beta) + \frac{W_{y0}+W_0}{\sqrt{1+W_0^2}}\cdot\frac{1}{\sin(\theta+\beta)}. \tag{9a}$$

The time when $W_y$ becomes zero is determined from (9a),

$$\theta = \arccos\frac{W_{y0}+W_0}{\sqrt{1+W_0^2}} - \arcsin\frac{1}{\sqrt{1+W_0^2}}, \tag{10}$$

after which the computation should be carried out by means of the equation for $\alpha_0 \leq \pi/2$, as has already been mentioned above. Seeking X, we analogously substitute

$$dY/d\theta = V_y = W_y \mp U_y,$$

and then integrate (9) and (9a). Consequently, for $\alpha_0 \leq \pi/2$ we obtain

$$Y = \ln\left[\frac{1-W_0}{2}(K^2\exp\theta + \exp(-\theta))\right] - \frac{2(W_0-W_{y0})}{\sqrt{1-W_0^2}}\operatorname{arctg}\frac{\exp\theta - 1}{K\exp\theta + 1/K} \mp U_y\theta. \tag{11}$$

For $W_0 > 1$, i.e., when the initial velocity is greater than the soaring velocity

$$Y = \ln\left[\frac{1-W_0}{2}(K^2\exp\theta + \exp(-\theta))\right] - \frac{2(W_0-W_{y0})}{\sqrt{-(1-W_0^2)}}\ln\left(\frac{1-i/K}{1+i/K}\cdot\frac{\exp\theta + i/K}{\exp\theta - i/K}\right) \mp U_y\theta. \tag{11a}$$

For $\alpha_0 > \pi/2$

$$Y = \ln\left[\frac{1}{\sin(\theta+\beta)}\cdot\frac{1}{\sqrt{1+W_0^2}}\right] + \frac{W_{y0}-W_0}{\sqrt{1+W_0^2}}\ln\left[\operatorname{tg}\frac{\theta+\beta}{2}(W_0+\sqrt{1+W_0^2})\right] \mp U_y\theta. \tag{11b}$$

The essential difference between the velocity $W_0$ and its components in the relationships presented above should be noted. In the latter $W_0$ is taken in absolute value, while $W_{y0}$ and $W_{x0}$ have signs corresponding to their directions.

As a comparison with computations performed on an electronic computer by using the system (1) shows, the dependences obtained above to determine the fundamental granule-motion parameters yield not more than a ±10% maximum error in the whole range of initial motion conditions.

## NOTATION

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| x | is the abscissa; |
| y | is the ordinate (measured from the top down); |
| **v** | is the particle velocity; |
| u | is the velocity of the medium; |
| w | is the relative particle velocity; |
| m | is the particle mass; |
| $\rho$ | is the density of the medium; |
| F | is the area of the Midelev section; |
| $\xi$ | is the frontal drag coefficient; |
| $\tau$ | is the time; |
| g | is the acceleration of gravity. |

## LITERATURE CITED

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